Phys 410
Fall 2013
Lecture #8 Summary
26 September, 2013

We began to discuss the physics of oscillations. Any system with a minimum in the potential energy landscape $U(x)$ can have small harmonic oscillations around the minimum in the potential. The Hooke's law spring constant is just the local curvature of the potential at the minimum: $k = d^2U/dx^2\big|_{x_0}$. A one-dimensional harmonic oscillator of mass m obeys the equation $m\ddot{x} = -kx$. Dividing through by m and defining the natural oscillation frequency $\omega_0 = \sqrt{k/m}$, the equation becomes $\ddot{x} = -\omega_0^2 x$. This equation can be solved in numerous ways, and the solutions can be written in several canonical forms, including:

1) $x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$
2) $x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$
3) $x(t) = A\cos(\omega t - \delta)$
4) $x(t) = \mathrm{Re}[Ae^{i(\omega t - \delta)}]$

All of these forms can be related to each other, as you will prove in homework.

We also considered the energy in simple harmonic motion. The total mechanical energy is $E = T + U = (m/2)\dot{x}^2 + (k/2)x^2$. Using form 3 above (for example), this can be written as $E = (k/2)A^2$, which is constant. The kinetic and potential energies are both varying with time as $\sin^2(\omega t - \delta)$ and $\cos^2(\omega t - \delta)$, respectively. They both oscillate between 0 and E twice per period of oscillation, and are exactly 180^o out of phase. The shuttling of energy back and forth between two different forms (in this case potential and kinetic) is a hallmark of simple harmonic oscillation, and resonance.

We considered un-driven damped oscillations produced by a damping force that is linear in velocity $m\ddot{x} + b\dot{x} + kx = 0$. This mechanical oscillator is a direct analog of the electrical oscillator made up of an inductor (L), resistor (R) and capacitor (C) in series. The charge on the capacitor plate $q(t)$ obeys the same equation: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$. The analogy is strong, as shown in the following table.

Mechanical Oscillator	**Electrical Oscillator**
Position x	Charge on capacitor plate q
Mass m	Inductance L
Damping b	Resistance R
Spring constant k	Inverse Capacitance $1/C$

Natural frequency $\omega_0 = [k/m]^{1/2}$	Natural frequency $\omega_0 = 1/[LC]^{1/2}$

Divide the mechanical equation through by mass m and define two important rates: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $2\beta \equiv b/m$, and $\omega_0^2 \equiv k/m$. We tried a solution of the form $x(t) = e^{rt}$, and found an auxiliary equation with solution $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. The general solution is $x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\ t}\right]$. The form of the solution depends critically on the relative size of the two rates β and ω_0.

1) Un-damped oscillator $\beta = 0$. The radical becomes $\sqrt{-\omega_0^2} = i\sqrt{\omega_0^2} = i\omega_0$, and the solution reverts to our previous results $x(t) = C_1 e^{i\omega_0\ t} + C_2 e^{-i\omega_0\ t}$.
2) Weak damping ($\beta < \omega_0$). The radical also produces a factor of "i", resulting in $x(t) = e^{-\beta t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$, with $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ a frequency lower than the un-damped natural frequency. This equation describes oscillatory motion under an exponentially damped envelope. The damping rate is β. One can re-write the solution as $x(t) = A\, e^{-\beta t}\cos(\omega_1 t - \delta)$.
3) Strong damping ($\beta > \omega_0$). In this case $\sqrt{\beta^2 - \omega_0^2}$ is real and the solution is $x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)\ t} + C_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_0^2}\right)\ t}$. This is a sum of two negative exponentials, one of which decays faster than the other – there is no oscillation.

We next considered a *driven* damped harmonic oscillator. We take the driving function to be harmonic in time at a new frequency called simply ω, which is an independent quantity from the natural frequency of the un-damped oscillator, called ω_0. The equation of motion is now $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$. We now employ a trick similar to that used to solve for the velocity of a charged particle in a uniform magnetic field. Consider the complementary problem of the same damped oscillator being driven by a force 90° out of phase, with solution $y(t)$: $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0\sin(\omega t)$. Now define a complex combination of the two unknown functions $z(t) = x(t) + iy(t)$. Combine the two equations in the form of "x-equation" + i "y-equation". This can be written more succinctly as $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. Note that the solution to the original problem can be found from $x(t) = Re[z(t)]$.

We now want to solve this equation: $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. We tried a solution of the form $z(t) = Ce^{i\omega t}$ and found this expression for the complex pre-factor: $C = \frac{f_0}{\omega_0^2 - \omega^2 + i2\beta\omega}$. We can write this complex quantity as a magnitude and phase as $C = Ae^{-i\delta}$, where A is the amplitude and δ is the phase, both real numbers. Solving for A and δ in terms of the oscillator parameters gives $A^2 = \frac{f_0^2}{\left(\omega_0^2-\omega^2\right)^2 + (2\beta\omega)^2}$, and $\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$. Finally, we can write the solution to the "z equation" as $z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}$.

The answer to the original problem is just the real part of this expression: $x(t) = Re[z(t)] = A\cos(\omega t - \delta)$, where ω is the frequency of the driving force. This represents the long-time persistent solution of the motion. It shows that the oscillator eventually adopts the same frequency as the driving force. In addition there is a solution to the homogeneous problem $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, which we solved before: $x_h(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\ t}\right]$. The full solution is the sum of the particular solution and the homogeneous solution. In the case of small loss ($\beta < \omega_0$) the full solution can be written as $x(t) = A\cos(\omega t - \delta) + A_{tr}\,\mathrm{e}^{-\beta t}\cos(\omega_1 t - \delta_{tr})$, where the first part is the particular solution and the second part is the transient (homogeneous) solution. We call it transient because of the $\mathrm{e}^{-\beta t}$ factor, which shows that the initial motion and initial conditions (specified by A_{tr} and δ_{tr}) will eventually die off and the persistent motion will dominate.